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Second-order effects in free convection

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The equations of conservation of momentum, energy and mass together with the equations of state are examined for free convection from a vertical paraboloid. A transformation due to Saville & Churchill is applied to the first- and second-order boundary-layer equations, which are then solved using series about the stagnation point, using asymptotic series far up the body and in between by a method due to Merk. The second-order outer inviscid flow is given in terms of infinite integrals as a solution of Laplace's equation in paraboloidal co-ordinates.

Eight second-order effects are distinguished, depending on longitudinal and transverse curvatures, the displacement flow, heat flux into the boundary layer and the variation of density, viscosity, thermometric conductivity and the coefficient of expansion with temperature. Expressions for the skin friction, heat-transfer coefficient and various flux thicknesses are obtained and a comparison of the second-order effects is made.

1. Introduction

Much attention in recent years has been focused on problems involving laminar free convection from a variety of body shapes. The usual starting point has been to make boundary-layer approximations under the assumption that the Grashof number G is large. The solution is derived using the first terms in asymptotic expansions of the momentum, energy and continuity equations as the Grashof number tends to infinity and for moderate Grashof numbers a second approximation is desirable.

For incompressible flows in the absence of body forces the second-order boundary-layer equations together with the appropriate matching and boundary conditions have been formulated by Van Dyke (1962*a*) using the method of matched asymptotic expansions. He notes that these equations are linear (as, indeed, are all higher order equations) and divides the second-order correction into five additive effects each capable of simply physical interpretation. These he labels as (i) longitudinal curvature, (ii) transverse curvature, (iii) displacement flow, (iv) external vorticity and (v) external gradient of stagnation temperature. In free convection (iv) and (v) are absent and we must include (iv) the heat flux and (v)–(viii) the variation of certain fluid properties with temperature. The latter may be taken to depend on χ, which is defined as $\beta(T_w - T_0)$, where β is the coefficient of thermal expansion and T_w and T_0 the temperatures of the body and the ambient fluid respectively. In some free convection flows χ is comparable with the second-order boundary-layer effects ($O(G^{-\frac{1}{4}})$) as illustrated in table 1. We assume that both curvatures are $O(1)$ everywhere and that non-continuum properties may be neglected.

A survey of second-order boundary layers in forced convection flows is given by Van Dyke (1969). Here the neglect of the body force simplifies the analysis by separating the momentum and energy equations and the inclusion of this term at the expense of the pressure gradient due to a free stream is the distinguishing feature of free convective flows. In the only work so far devoted to thermally induced second-order boundary layers, Yang & Jerger (1964) and Clarke (1973) discuss free convection from a vertical flat plate where the only second-order effect present is that due to the displacement flow. Variable fluid properties are not considered but Poots & Raggett (1967) have discussed them in forced convection. The effect of dissipation in free convection (which we shall neglect) has been treated by Gebhart (1962).

In this study we consider a flow in which all five second-order effects are present and choose, as a simple case of a body possessing both longitudinal and transverse curvature, a paraboloid of revolution. This is placed with its axis vertical and is assumed to extend upwards to infinity. As in all free convection flows there is no obvious velocity scale and we define $U_0^2 = g\beta l(T_w - T_0)$, where l is a typical length scale (say, the radius of curvature at the station point). The Grashof number is then $U_0^2 l/\nu^2$.

The first- and second-order boundary-layer equations are formulated in a similar way to that of Van Dyke (1962*a*) and are solved in §§3 and 5 respectively. Series solutions are obtained near the stagnation point and far up the body after application of a transformation due to Saville & Churchill (1967) which is analogous to Görtler's (1957). This has the advantage that the velocity and temperature profiles change slowly as we proceed along the body and we are therefore able to find the solution where neither series holds by a method due to Merk (1959) which is much faster than the usual step-by-step integration procedure.

The second-order effect due to the displacement flow involves the second-order outer (inviscid) flow. There is, of course, no first-order outer flow. It is here that the elliptic nature of the Navier–Stokes equations reasserts itself, having been lost in the boundary-layer approximations, which yield parabolic equations. In §4 we discuss a solution of Laplace's equation in paraboloidal co-ordinates in which the tangential component of the velocity at the surface is expressed in terms of infinite integrals involving the second-order normal component.

Finally, in §6, expressions for the skin friction, heat-transfer coefficients and the flux thicknesses are obtained and the results discussed.

2. Equations of motion

Our co-ordinate system (s, N, ω) is defined with reference to cylindrical polar co-ordinates (r, z, ω) with the origin at the stagnation point and z measured vertically upwards; s is measured along the body and N normal to it. A length element is given by

$$(dl)^2 = (1+\kappa N)^2 (ds)^2 + (dN)^2 + (1+N\sin\theta)^2 (d\omega)^2,$$

where $\kappa = -(d^2r/ds^2)\operatorname{cosec}\theta$ is the curvature in the s, N plane and θ is the angle between the tangent and horizontal.

The momentum, energy and continuity equations for a compressible fluid are

$$\rho\frac{DU_i}{Dt} = \rho X_i - \frac{\partial}{\partial x_i}\left(p + \tfrac{2}{3}\mu\frac{\partial U_j}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\right\}, \tag{2.1}$$

$$\rho\frac{DI}{Dt} = \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) + \frac{Dp}{Dt} + \Phi, \tag{2.2}$$

$$D\rho/Dt + \rho\,\partial U_i/\partial x_i = 0, \tag{2.3}$$

in the usual notation. These are supplemented by equations of state for ρ, μ, k and I. For a homogeneous fluid I, ρ, μ and k are functions of p and T with $(\partial I/\partial T)_p = c_p$. For a gas we assume that $c_p =$ constant, $I = c_pT$, $p = R\rho T$, $\mu = \mu(T)$, $k = k(T)$ and $\sigma \equiv \mu c_p/k =$ constant; for a liquid $\rho = \rho(T)$ and $I = I(T)$, which means that $c_p = I'(T)$, $\mu = \mu(T)$, $k = k(T)$ and $\sigma = \sigma(T)$. The boundary conditions are

$$T = T_w, \quad \mathbf{U} = 0 \quad \text{at} \quad N = 0; \quad T \to T_0, \quad \mathbf{U} \to 0 \quad \text{as} \quad N \to \infty.$$

We write $p = p_s + p_d$, $\rho = \rho_s + \rho_d$ and $T = T_s + T_d$, where the subscripts s and d denote static and dynamic variables respectively. When the fluid is static $(U_i \equiv 0)$ we have $dp_s/dz = -g\rho_s$, and $k(T_s)\,dT_s/dz =$ constant. For a homentropic gas $T_s(z) = T_s(0) - gz/c_p$. A characteristic velocity is defined by $U_0 = (\beta g l T_1)^{\frac{1}{2}}$, where the local coefficient of expansion $\beta = T_0^{-1}$ for a gas and $\beta = -\rho'(T_0)\rho_0^{-1}$ for a liquid. Also $T_1 = T_w - T_0$ is a typical temperature difference and l is a typical length (say, the radius of curvature at the origin). We non-dimensionalize by writing

$$\mathbf{x} = l\bar{\mathbf{x}}, \quad \mathbf{U} = U_0\bar{\mathbf{U}}, \quad T = T_s(z) + T_1\bar{T}, \quad \rho = \rho_s(z) - \rho_0\chi\bar{\rho},$$

$$p = p_s(z) + \chi\rho_0 g l\bar{p}, \quad c_p = c_0\bar{c}_p, \quad \mu = \mu_0\bar{\mu}, \quad k = k_0\bar{k},$$

where $\chi = \beta T_1$ and ρ_0, c_0, μ_0 and k_0 are stagnation values. Assuming that the stagnation temperature gradient may be neglected, i.e. $\beta l(dT_s/dz)_0 \ll \chi$, we may take $T_s(z) = T_s(0) = T_0$ and $\rho_s = \rho_s(0) = \rho_0$, so that the equations of state may be written as

$$\rho = \rho_0(1 - \chi\bar{\rho}),$$

$$\left.\begin{aligned} \bar{\rho} &= \bar{T} + \beta_1\chi\bar{T}^2, & \bar{\mu} &= 1 + \chi\mu_1\bar{T}, \\ \bar{k} &= 1 + \chi k_1\bar{T}, & \bar{c}_p &= 1 + \chi c_1\bar{T}, \end{aligned}\right\} \tag{2.4}$$

where β_1, μ_1, k_1 and c_1 are known coefficients which are assumed to be $O(1)$. This necessitates avoiding temperatures near which β is small for then

$$\beta_1 = O(\chi^{-1}) \gg 1$$

and we must therefore exclude discussion of effects such as the anomalous expansion of water. For a gas we take $\beta_1 = -1$ and $c_1 = 0$, and we may ignore terms in χ for a liquid (see table 1), so that hereafter $\bar{c}_p = 1$.

The equation of continuity reduces to

$$\partial\bar{U}_i/\partial x_i = \chi\,D\bar{\rho}/Dt.$$

	l (cm)	T_1 (°C)	G	$G^{-\frac{1}{4}}$	χ	ϵ
Air ($T_w = 10$ °C)	2	1	10^6	0·03	0·004	$3{\cdot}0\times10^{-5}$
	1	10	10^6	0·03	0·04	$1{\cdot}5\times10^{-5}$
	0·5	100	10^6	0·03	0·4	$0{\cdot}7\times10^{-5}$
Water	80	1	10^6	0·03	0·0006	$1{\cdot}2\times10^{-5}$
($T_w = 10$ °C)	40	10	10^6	0·03	0·006	$0{\cdot}6\times10^{-5}$
	8	10	10^4	0·1	0·006	$1{\cdot}0\times10^{-6}$
Mercury ($T_w = 10$ °C)	1	0·1	10^6	0·03	0·0002	$6{\cdot}0\times10^{-6}$

TABLE 1. Physical parameters

The temperature rise due to the dissipation term Φ is $O(U_0^2/c_p)$, which is $O(\epsilon T_1)$, where $\epsilon = \beta g l/c_p$, and in the energy equation $Dp/Dt = O(\epsilon/\chi)$. We hereafter assume that $\epsilon \ll \chi^2$ (see table 1) and neglect dissipation effects. The simplified equations of motion are

$$(1-\chi T)\frac{\partial U_i}{Dt} = (T+\beta_1\chi T^2)\hat{g}_i - \frac{\partial p}{\partial x_i} + G^{-\frac{1}{2}}\left\{\frac{\partial^2 U_i}{\partial x_j^2} + \chi\left[\frac{1}{3}\frac{\partial}{\partial x_i}\frac{DT}{Dt} + \mu_1\frac{\partial T}{\partial x_j}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)\right]\right\}, \tag{2.5}$$

$$\sigma(1-\chi T)\frac{DT}{Dt} = G^{-\frac{1}{2}}\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right), \tag{2.6}$$

$$\partial U_i/\partial x_i = \chi DT/Dt, \tag{2.7}$$

on dropping the bars, where $T \equiv I$, $\hat{\mathbf{g}}$ is a unit vector in the z direction and

$$G = \beta l^3 g T_1 \rho_0^2/\mu_0^2 \quad \text{(Grashof number)},$$

$$\sigma = c_0\mu_0/k_0 \quad \text{(Prandtl number)}.$$

The boundary conditions in non-dimensional form are

$$T = 1, \quad \mathbf{U} = 0 \quad \text{at} \quad N = 0; \quad T \to 0, \quad \mathbf{U} \to 0 \quad \text{as} \quad N \to \infty. \tag{2.8}$$

2.1. *Inner and outer expansions*

For $G \gg 1$ we seek an outer solution of the form

$$\mathbf{U} = \mathbf{U}_0 + G^{-\frac{1}{4}}\mathbf{U}_1 + \ldots,$$

and similarly for T and p. Equations (2.5)–(2.7) and boundary conditions (2.8) then give

$$\left.\begin{aligned} &\mathbf{U}_0 \equiv 0, \quad T_0 \equiv 0, \\ &(\mathbf{U}_1.\nabla)\mathbf{U}_1 + \nabla P_1 = -T_1\hat{g}, \\ &(\mathbf{U}_1.\nabla)T_1 = 0, \quad \nabla.\mathbf{U}_1 = 0, \end{aligned}\right\} \tag{2.9}$$

which means that $T_1 \equiv 0$ and $\mathbf{U}_1$ derives from a velocity potential which satisfies Laplace's equation.

The boundary conditions at the body cannot be satisfied completely and we look for boundary-layer solutions with velocity components $(u, v, 0)$ by writing

$N = G^{-\frac{1}{4}}n$ and $V = G^{-\frac{1}{4}}v$. Equations (2.5)–(2.7) then becomes (Rosenhead 1963, p. 130)

$$(1-\chi T)\left\{\frac{u}{h_1}\frac{\partial u}{\partial s}+\frac{v}{h_2}\frac{\partial u}{\partial n}+\frac{uv}{h_1h_2}\frac{\partial h_1}{\partial n}\right\}+\frac{1}{h_1}\frac{\partial p}{\partial s} = T(1+\beta_1\chi T)\sin\theta-\frac{G^{\frac{1}{4}}}{h_2h_3}\frac{\partial}{\partial n}(\mu h_3\zeta_3), \tag{2.10a}$$

$$(1-\chi T)\left\{\frac{u}{h_1}\frac{\partial v}{\partial s}+\frac{v}{h_2}\frac{\partial v}{\partial n}-\frac{u^2}{h_1h_2}\frac{\partial h_1}{\partial n}G^{\frac{1}{4}}\right\}+\frac{G^{\frac{1}{2}}}{h_2}\frac{\partial p}{\partial n}$$
$$= -G^{-\frac{1}{4}}\left\{T(1+\beta_1\chi T)\cos\theta-\frac{1}{h_1h_3}\frac{\partial}{\partial s}(\mu h_3\zeta_3)\right\}, \tag{2.10b}$$

$$(1-\chi T)\left\{\frac{u}{h_1}\frac{\partial T}{\partial s}+\frac{v}{h_2}\frac{\partial T}{\partial n}\right\} = \sigma^{-1}\left\{\frac{G^{-\frac{1}{4}}}{h_1}\frac{\partial}{\partial s}+\frac{1}{h_2}\frac{\partial}{\partial n}\right\}\left\{G^{-\frac{1}{4}}\frac{k}{h_1}\frac{\partial T}{\partial s}+\frac{k}{h_2}\frac{\partial T}{\partial n}\right\}, \tag{2.10c}$$

$$\partial(h_3u)/\partial s+\partial(h_1h_3v)/\partial n = 0, \tag{2.10d}$$

where $\quad h_1 = 1+G^{-\frac{1}{4}}\kappa n, \quad h_2 = 1, \quad h_3 = r+G^{-\frac{1}{4}}n\sin\theta$

and $\quad \zeta_3 = -G^{-\frac{1}{4}}(\partial u/\partial n+G^{-\frac{1}{4}}\kappa u/h_1-G^{-\frac{1}{2}}\partial v/\partial s).$

Again we expand in inverse powers of G, so that

$$\mathbf{u} = \mathbf{u}_0+G^{-\frac{1}{4}}\mathbf{u}_1+\dots,$$

and similarly for T and p. Then we have

$$u_0\frac{\partial u_0}{\partial s}+v_0\frac{\partial u_0}{\partial n}+\frac{\partial p_0}{\partial s} = T_0\sin\theta+\frac{\partial^2u_0}{\partial n^2}, \tag{2.11a}$$

$$\partial p_0/\partial n = 0, \tag{2.11b}$$

$$u_0\frac{\partial T_0}{\partial s}+v_0\frac{\partial T_0}{\partial n} = \frac{1}{\sigma}\frac{\partial^2T_0}{\partial n^2}, \tag{2.11c}$$

$$\partial(ru_0)/\partial s+\partial(rv_0)/\partial n = 0, \tag{2.11d}$$

where $u_0 = v_0 = T_0$ at $n = 0$ and as $n\to\infty$ we match with the outer flow to obtain $u_0, T_0\to 0$. These are the familiar boundary-layer equations, in which we have made the Boussinesq approximation $\chi \ll 1$. Equating terms in $G^{-\frac{1}{4}}$ in (2.10) and writing $\chi = G^{-\frac{1}{4}}\overline{\chi}$ we obtain the second-order boundary-layer equations

$$u_1\frac{\partial u_0}{\partial s}+v_1\frac{\partial u_0}{\partial u}+u_0\frac{\partial u_1}{\partial s}+v_0\frac{\partial u_1}{\partial n}-T_1\sin\theta-\frac{\partial^2u_1}{\partial u^2}+\frac{\partial p_1}{\partial s}$$
$$= \kappa nu_0\frac{\partial u_0}{\partial s}-\kappa\frac{\partial u_0}{\partial n}+\kappa u_0v_0+\frac{\sin\theta}{r}\frac{\partial u_0}{\partial n}$$
$$+\overline{\chi}\left\{\beta_1T_0^2+T_0\left(u_0\frac{\partial u_0}{\partial s}+v_0\frac{\partial u_0}{\partial n}\right)+\mu_1\left(T_0\frac{\partial^2u_0}{\partial n^2}+\frac{\partial T_0}{\partial n}\frac{\partial u_0}{\partial n}\right)\right\}, \tag{2.12a}$$

$$\partial p_1/\partial n = \kappa u_0^2-T_0\cos\theta, \tag{2.12b}$$

$$u_1\frac{\partial T_0}{\partial s}+v_1\frac{\partial T_0}{\partial n}+u_0\frac{\partial T_1}{\partial s}+v_0\frac{\partial T_1}{\partial n}-\frac{1}{\sigma}\frac{\partial^2T_1}{\partial n^2} = \kappa nu_0\frac{\partial T_0}{\partial s}+\frac{1}{\sigma}\frac{\partial T_0}{\partial n}\left(\frac{\sin\theta}{r}+\kappa\right)$$
$$+\overline{\chi}\left\{T_0\left(u_0\frac{\partial T_0}{\partial s}+v_0\frac{\partial T_0}{\partial n}\right)+k_1\left(\frac{\partial^2T_0}{\partial n^2}+\left(\frac{\partial T_0}{\partial n}\right)^2\right)\right\}, \tag{2.12c}$$

$$\frac{\partial}{\partial s}(ru_1+n\sin\theta\, u_0-\overline{\chi}ru_0T_0)+\frac{\partial}{\partial n}(rv_1+v_0(rn\kappa+n\sin\theta-\overline{\chi}rT_0)) = 0. \tag{2.12d}$$

The boundary conditions are $u_1 = v_1 = T_1 = 0$ at $n = 0$ and matching with the second-order outer flow gives $T_1 \to 0$ and $u_1 \to U_1(s, 0)$ as $n \to \infty$. The boundary conditions for the second-order outer flow as $N \to 0$ obtained by matching with the first-order boundary layer as $n \to \infty$ are

$$V_1(s,0) = \lim_{n\to\infty} v_0(s,n), \quad T_1(s,0) = \lim_{n\to\infty} T_1(s,n) = 0.$$

3. The first-order boundary layer

Equation (2.11 b) means that the pressure is constant across the boundary layer and takes its value at the outer edge. Hence $\partial p_0/\partial s = 0$. Equation (2.11 d) is satisfied directly by the introduction of a stream function ψ_0, defined by

$$u_0 = \frac{1}{r}\frac{\partial \psi_0}{\partial n}, \quad v_0 = -\frac{1}{r}\frac{\partial \psi_0}{\partial s}.$$

We then have

$$\psi_{0n}\left(\psi_{0ns} - \frac{1}{r}\frac{dr}{ds}\psi_{0n}\right) - \psi_{0s}\psi_{0nn} = r^2 T_0 \sin\theta + r\psi_{0nnn}, \tag{3.1a}$$

$$\psi_{0n} T_{0s} - \psi_{0s} T_{0n} = (r/\sigma)\, T_{0nn}, \tag{3.1b}$$

$$\psi_{0s}(s,0) = \psi_{0n}(s,0) = 0, \quad T_0(s,0) = 1, \tag{3.1c}$$

$$\lim_{n\to\infty} \psi_{0n}(s,n) = \lim_{n\to\infty} T_0(s,n) = 0, \tag{3.1d}$$

where subscripts s and n denote differentiation.

Solutions are best obtained by dividing the range of integration in the s direction into three parts. Near the stagnation point ψ_0 and T_0 are expanded in power series in s with coefficients functions of n which satisfy certain ordinary differential equations. For $s \gg 1$ asymptotic series are obtained in inverse powers of s, while in the middle region (3.1) usually must be integrated numerically. For isothermal streaming flows Görtler (1957) found that a transformation of the variables gave improved series solutions and Saville & Churchill (1967) have provided a similar transformation for flows with a body force. It is this that we use here.

We first use Mangler's transformation to reduce (3.1) to a form similar to that for planar flows. We write

$$\bar{s} = \int_0^s r^2\, ds, \quad \bar{n}, = rn,$$

$$\overline{\psi}_{0\bar{n}} = \frac{1}{r}\psi_{0n}, \quad \overline{\psi}_{0\bar{s}} = \frac{1}{r^2}\psi_{0s} - \frac{dr}{ds}\frac{n}{r^3}\psi_{0n}.$$

Then (3.1) become on dropping the bars

$$\left.\begin{aligned} \psi_{0n}\psi_{0ns} - \psi_{0s}\psi_{0nn} &= (\sin\theta)/r^2 T_0 + \psi_{0nnn}, \\ T_{0nn} + \sigma(\psi_{0s}T_{0n} - \psi_{0n}T_{0s}) &= 0. \end{aligned}\right\} \tag{3.2}$$

Using the Saville–Churchill transformation

$$\xi = \int_0^s \left(\frac{\sin\theta}{r^2}\right)^{\frac{1}{3}} ds, \quad \eta = \left(\frac{4\xi}{3}\right)^{-\frac{1}{4}} \left(\frac{\sin\theta}{r^2}\right)^{\frac{1}{3}} n,$$

$$\psi_0 = (\tfrac{4}{3}\xi)^{\frac{3}{4}} F_0(\xi,\eta), \quad T_0 = T_0(\xi,\eta),$$

equations (3.2) become

$$\left.\begin{aligned} \tfrac{4}{3}\xi\,(F_{0\eta}F_{0\xi\eta}-F_{0\xi}F_{\eta\eta})-F_0F_{0\eta\eta}+\tfrac{4}{3}K(\xi)\,F_{0\eta}^2 &= T_0+F_{0\eta\eta\eta}, \\ \tfrac{4}{3}\xi\,(F_{0\eta}T_{0\xi}-F_{0\xi}T_{0\eta})-F_0T_{0\eta} &= \sigma^{-1}T_{0\eta\eta}, \end{aligned}\right\} \tag{3.3}$$

where $F_0 = F_{0\eta} = 0$ and $T_0 = 1$ at $\eta = 0$ and $F_{0\eta}, T_0 \to 0$ as $\eta \to \infty$. $K(\xi)$ corresponds to Görtler's 'principal function' and is defined by

$$K(\xi) = \frac{1}{2}+\frac{\xi}{3}\frac{r^2}{\sin\theta}\frac{d}{d\xi}\left(\frac{\sin\theta}{r^2}\right).$$

The shape of the body enters these equations only through the coefficient $K(\xi)$ and for particular body shapes for which $K(\xi)$ is constant we may obtain similarity solutions. Here $K(\xi)$ is not constant and we seek solutions for $\xi \ll 1$ by expanding in powers of ξ. Thus

$$K(\xi) = \sum_{j=0}^{\infty} K_j(\xi/K_0)^{\alpha j},$$

where for round-nosed bodies $K_0 = \frac{3}{8}$ and $\alpha = \frac{3}{4}$, and in particular for a paraboloid $K_1 = -\frac{3}{28}$, $K_2 = \frac{141}{980}$. We expand F_0 and T_0 in powers of $\xi^{\frac{3}{4}}$, i.e.

$$F_0 = F_{00}+K_1F_{01}(\xi/K_0)^{\frac{3}{4}}+(K_2F_{02}+K_1^2F_{011})\,(\xi/K_0)^{\frac{3}{2}}+\ldots,$$
$$T_0 = T_{00}+K_1T_{01}(\xi/K_0)^{\frac{3}{4}}+(K_2T_{02}+K_1^2T_{011})\,(\xi/K_0)^{\frac{3}{2}}+\ldots,$$

then equating coefficients of $\xi^{\alpha j}$ in (3.3) gives

$$\left.\begin{aligned} F_{00}'''+F_{00}F_{00}''-\tfrac{1}{2}F_{00}'^2+T_{00} &= 0, \\ T_{00}''+\sigma F_{00}T_{00}' &= 0, \end{aligned}\right\} \tag{3.4}$$

where $\qquad F_{00}(0) = F_{00}'(0) = F_{00}'(\infty) = T_{00}(\infty) = 0, \quad T_{00}(0) = 1,$

and a sequence of linear equations for the higher order coefficients.†

For $\xi \gg 1$ we find that for a paraboloid

$$K(\xi) \sim \tfrac{3}{10}+\bar{K}_1\xi^{-1}+\bar{K}_2\xi^{-\frac{6}{5}}+O(\xi^{-\frac{9}{5}}),$$

where $\bar{K}_1 = 0{\cdot}052998$ and $\bar{K}_2 = -(\frac{3}{10})^{\frac{6}{5}}\frac{3}{20}$. This suggests that we look for asymptotic solutions of (3.3) of the form

$$F_0(\xi,\eta) = \bar{F}_{00}+\bar{K}_1\bar{F}_{01}\xi^{-1}+\bar{K}_2\bar{F}_{02}\xi^{-\frac{6}{5}}+\Sigma\beta_{\lambda_i}\bar{F}_{0\lambda_i}\xi^{-\lambda_i}+O(\xi^{-\frac{9}{5}}),$$

and similarly for T_0, where the λ_i are the possible eigenvalues of (3.3) and the β_{λ_i} are scaling factors chosen so that $\bar{F}_{0\lambda_i}''(0) = 1$. Equating coefficients of powers of ξ we obtain

$$\left.\begin{aligned} \bar{F}_{00}'''+\bar{T}_{00}+\bar{F}_{00}\bar{F}_{00}''-\tfrac{2}{5}\bar{F}_{00}'^2 &= 0, \\ \bar{T}_{00}''+\sigma\bar{F}_{00}\bar{T}_{00}' &= 0, \end{aligned}\right\} \tag{3.5}$$

where $\qquad \bar{F}_{00}(0) = \bar{F}_{00}'(0) = \bar{F}_{00}'(\infty) = \bar{T}_{00}(\infty) = 0, \quad \bar{T}_{00}(0) = 1,$

and a sequence of linear equations for the further unknown coefficients. β_{λ_1} is found by comparing the velocity and temperature profiles with those obtained

† These and subsequent sets of linear equations for higher order coefficients are given in longer versions of §§ 3 and 5 available on request from the J.F.M. Editorial Office, Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge CB3 9EW.

in the following section for some value of ξ where both solutions are valid. For $\sigma = 1$ we obtain $\beta_{\lambda_1} = -0{\cdot}052 \pm 0{\cdot}001$ by comparing profiles at $\xi = 10$ and numerical integration of the linear equations for $\overline{F}_{0\lambda_1}$ gives $\lambda_1 = 0{\cdot}962914$ for $\sigma = 1$.

3.1. *Solution in the region where neither series holds*

In order to calculate the outer flow we need solutions of the boundary-layer equations at all points along the body. Van Dyke (1964), treating the flow past a parabolic cylinder, achieved this by manipulation of series in s, while Clark & Watson (1971) were able to use similarity solutions. Neither approach seems likely to succeed here and we must turn our attention to numerical methods. Terrill (1960), Merkin (1969) and Switzer (1969) have developed step-by-step finite-difference integration procedures to continue the series solution for $\xi \ll 1$ into the region of validity of the series for $\xi \gg 1$. Large but sparse matrices must be inverted and, because the equations are nonlinear, iteration is necessary at each step; the step length along the body is very small at first but is systematically increased later.

A much faster method due to Merk (1959) is applicable whenever $\xi K'(\xi)$ is small, as it is here. This is equivalent to assuming that $K(\xi)$ is almost constant, i.e. the departure from similarity solutions is small. In fact, the first term in the expansion (3.7) below is a local similarity solution and the subsequent terms are corrections to it. We first change independent variables from (ξ, η) to (K, η) and (3.3) becomes

$$\left.\begin{aligned} F_{0\eta\eta\eta} + T_{0\eta} + F_0 F_{0\eta\eta} - \tfrac{4}{3}K(\xi)\, F_{0\eta}^2 &= \tfrac{4}{3}\xi K'(\xi)\,(F_{0\eta} F_{0K\eta} - F_{0K} F_{0\eta\eta}),\\ T_{0\eta\eta} + \sigma F_0 T_{0\eta} &= \sigma \times \tfrac{4}{3}\xi K'(\xi)\,(F_{0\eta} T_{0K} - F_{0K} T_{0\eta}), \end{aligned}\right\} \tag{3.6}$$

where $$F_0(0) = F_{0\eta}(0) = F_{0\eta}(\infty) = T_0(\infty) = 0, \quad T_0(0) = 1.$$

F_0 may then be expanded as

$$F_0 = F_{00}(K,\eta) + \tfrac{4}{3}\xi K'(\xi)\, F_{01}(K,\eta) + (\tfrac{4}{3}\xi)^2\,[K''(\xi)\, F_{021}(K,\eta) + K'^2(\xi)\, F_{022}(K,\eta)] + \ldots, \tag{3.7}$$

and similarly for T_0. Substituting into (3.6) we get

$$\left.\begin{aligned} F'''_{00} + F_{00}F''_{00} + T_{00} - \tfrac{4}{3}KF'^2_{00} &= 0,\\ T''_{00} + \sigma F_{00} T'_{00} &= 0, \end{aligned}\right\} \tag{3.8}$$

where $$F_{00}(0) = F'_{00}(0) = F'_{00}(\infty) = T_{00}(\infty) = 0, \quad T_{00}(0) = 1.$$

The linear equations for the higher order coefficients are similar to Merk's but differ because of his omission of certain terms in his equations (25) and (26).

It is now a straightforward matter to obtain the solution for any ξ by solving a system of ordinary differential equations in which K is a known parameter. In fact initial guesses for the 'shooting' method used here are furnished by the solution at the previous value of ξ and convergence is rapid. Comparison with results obtained by step-by-step integration show that Merk's method is sufficiently accurate for our present purposes for $0{\cdot}1 \leqslant \xi \leqslant 10{\cdot}0$. Outside this range the series solutions are to be used.

4. The outer flow

The outer flow is irrotational and the velocity potential ϕ satisfies Laplace's equation. We transform to paraboloidal co-ordinates (x,y) defined by $z = \frac{1}{2}(x^2-y^2)$, $r = xy$, so that $y = 1$ is the surface of the paraboloid $z = \frac{1}{2}(r^2-1)$ and on $y = 1$, $x = r$. ϕ satisfies $\nabla^2\phi = 0$ in $y > 1$ and the component of velocity in the y direction, $V_1(x,y)$, satisfies the condition at the surface

$$\lim_{y\to 1+} V_1(x,y) = \lim_{y\to 1+} (1+x^2)^{-\frac{1}{2}}\,\partial\phi/\partial y = \lim_{\eta\to\infty} v_0(\xi,\eta) = v_0(\xi). \tag{4.1}$$

$\nabla^2\phi = 0$ takes the form

$$y\frac{\partial}{\partial x}\left(x\frac{\partial\phi}{\partial x}\right) + x\frac{\partial}{\partial y}\left(y\frac{\partial\phi}{\partial y}\right) = 0$$

and a solution bounded as $x\to 0$ and tending to zero as $y\to\infty$ is

$$\phi = AJ_0(kx)\,K_0(ky),$$

where A and k are constants. We therefore seek to represent the solution by the general form

$$\phi = \int_0^\infty f(k)\,J_0(kx)\,K_0(ky)\,dk, \tag{4.2}$$

so that

$$V_1(x_1,y) = (1+x^2)^{-\frac{1}{2}}\int_0^\infty f(k)\,kK_1(ky)\,J_0(kx)\,dk.$$

By Hankel's inversion formula with $y = 1$ we get

$$\begin{aligned} f(k)\,K_1(k) &= -\int_0^\infty x_1(1+x_1^2)^{\frac{1}{2}}\,V_1(x_1,1)\,J_0(kx_1)\,dx_1 \\ &= -\int_0^\infty x_1(1+x_1^2)^{\frac{1}{2}}\,v_0(x_1)\,J_0(kx_1)\,dx_1 \quad \text{by (4.1).} \end{aligned}$$

The tangential component of velocity at the surface is given by

$$\begin{aligned} (1+x^2)^{\frac{1}{2}}\,U_1(x) &= \lim_{y\to 1+} -\int_0^\infty kf(k)\,J_1(kx)\,K_0(ky)\,dk \\ &= \lim_{y\to 1+}\int_0^\infty \frac{kK_0(ky)}{K_1(k)}\left\{\int_0^\infty x_1(1+x_1^2)^{\frac{1}{2}}\,v_0(x_1)\,J_0(kx_1)\,dx_1\right\} J_1(kx_1)\,dk \\ &= \int_0^\infty x_1(1+x_1^2)^{\frac{1}{2}}\,v_0(x_1)\left\{\lim_{y\to 1+}\int_0^\infty \frac{k\,K_0(ky)}{K_1(k)}\,J_0(kx_1)\,J_1(kx)\,dk\right\} dx_1. \end{aligned} \tag{4.3}$$

The limit process is necessary because $K_0(ky)/K_1(k) \sim \exp\{-(y-1)\,k\}$ as $k\to\infty$ and so the inner integral in (4.3) does not converge for $y = 1$. $\lim\limits_{y\to 1+} K_0(ky)/K_1(k)$ is equivalent to $\lim\limits_{\delta\to 0+} e^{-k\delta}$ and we may write

$$\begin{aligned} \lim_{y\to 1+}\int_0^\infty k\frac{K_0(ky)}{K_1(k)}\,J_0(kx_1)\,J_1(kx)\,dk &= \int_0^\infty k\left(\frac{K_0(k)}{K_1(k)}-1\right) J_0(kx_1)\,J_1(kx)\,dk \\ &\quad + \lim_{\delta\to 0+}\int_0^\infty e^{-k\delta}\,kJ_0(kx_1)\,J_1(kx)\,dk = f(x,x_1)+g(x,x_1), \quad \text{say.} \end{aligned} \tag{4.4}$$

Also $$\int_0^\infty e^{-k\delta} J_0(kx)\, J_0(kx_1)\, dk = \frac{2K(\overline{m})}{\pi(\delta^2+(x+x_1)^2)^{\frac{1}{2}}}$$

(Watson 1962, equation (13.22)), where $\overline{m}^2 = 4xx_1[\delta^2+(x+x_1)^2]^{-1}$ and $K(m)$ is the elliptic integral of the first kind, so that

$$g(x,x_1) = -\partial\{2K(m)/\pi(x+x_1)\}/\partial x, \tag{4.5}$$

where $m^2 = 4xx_1(x+x_1)^{-2}$. As $x \to x_1$, $m \to 1$ and $K(m) \sim -\log|x-x_1|$. Hence

$$g(x,x_1) \sim [\pi x_1(x-x_1)]^{-1} \quad \text{as} \quad x \to x_1.$$

From (4.3)–(4.5) we obtain

$$(1+x_1^2)^{\frac{1}{2}}\, U_1(x) = \int_0^\infty x_1(1+x_1^2)^{\frac{1}{2}} f(x,x_1)\, dx_1 - \int_0^\infty \{x_1(1+x_1^2)^{\frac{1}{2}}\, v_0(x_1) - x(1+x^2)^{\frac{1}{2}}\, v_0(x)\}$$
$$\times \frac{\partial}{\partial x}\left[\frac{2K(m)}{\pi(x+x_1)}\right] dx_1 + x(1+x^2)^{\frac{1}{2}}\, v_0(x) \lim_{\delta,\epsilon\to 0} \int_0^\infty e^{-\epsilon x_1} \int_0^\infty e^{-k\delta}\, kJ_0(kx_1)\, J_1(kx)\, dk\, dx_1, \tag{4.6}$$

where the second integral on the right takes the value $(\pi x)^{-1} d[x(1+x^2)^{\frac{1}{2}}\, v_0(x)]/dx$ at $x_1 = x$ and a further converging factor $e^{-\epsilon x_1}$ has been introduced into the last integral. By inverting the order of integration this term may be shown to be equal to $(1+x^2)^{\frac{1}{2}}\, v_0(x)$.

The expression contained in (4.4) for $f(x,x_1)$ converges slowly and from a computational point of view a better expression is obtained by distorting the path of integration from the real to the imaginary axis. Thus, for $x > x_1$ we may write

$$\begin{aligned} f(x,x_1) &= \frac{1}{2}\int_0^\infty k\left[\frac{K_0(k)}{K_1(k)} - 1\right][H_1^{(1)}(kx) + H_1^{(2)}(kx)]\, J_0(kx_1)\, dk \\ &= \frac{1}{2}\int_0^\infty (it)\left[\frac{K_0(it)}{K_1(it)} - 1\right] H_1^{(1)}(ixt)\, J_0(ix_1 t)\, dt \\ &\qquad + \frac{1}{2}\int_0^\infty (-it)\left[\frac{K_0(-it)}{K_1(-it)} - 1\right] H_1^{(2)}(-ixt)\, J_0(-ix_1 t)\, (-i)\, dt \\ &= -\frac{4}{\pi^2}\int_0^\infty \left[\frac{\pi t}{2} - \frac{1}{J_1^2(t) + Y_1^2(t)}\right] K_1(xt)\, I_0(x_1 t)\, dt. \end{aligned}$$

Similarly, for $x < x_1$ we obtain

$$f(x,x_1) = \frac{4}{\pi^2}\int_0^\infty \left[\frac{\pi t}{2} - \frac{1}{J_1^2(t) + Y_1^2(t)}\right] I_1(xt)\, K_0(x_1 t)\, dt,$$

and hence $f(x,x+) - f(x+,x) = 2x$. These integrands behave like

$$t^{-2}\exp(-|x-x_1|t) \quad \text{as} \quad t \to \infty$$

and can easily be evaluated numerically. Asymptotic expansions of the integrands in (4.6) as $x_1 \to \infty$ depend on x/x_1 and have been obtained sufficiently accurately for the upper limits to be replaced by max $(10, 3x)$ and max $(10, 7x)$ respectively to give results correct to four places of decimals. Sample results for $U_1(x)$ as well as $V_1(x, 1)$ are shown in table 2 for $\sigma = 1$.

ξ	$x\ (\equiv r)$	V_1	U_1
0·01	0·2557	−1·4197	0·1111
0·1	0·5954	−1·3518	0·2203
0·2	0·7626	−1·3086	0·2561
0·5	1·0495	−1·2302	0·2728
1·0	1·3275	−1·1571	0·2692
2·0	1·6697	−1·0762	0·2600
5·0	2·2437	−0·9646	0·2215
10·0	2·7925	−0·8814	0·1834
20·0	3·4648	−0·8018	0·1497
50·0	4·5925	−0·7044	0·1191
100·0	5·6730	−0·6372	0·0928

TABLE 2. The outer solution

An asymptotic solution for ϕ as $x \to \infty$ may be found by taking spherical polar co-ordinates (R, θ, ω) with the origin at the stagnation point and writing $V_R = \partial\phi/\partial R$ and $V_\theta = R^{-1}\partial\phi/\partial\theta$. We then have $V_1 \sim -V_\theta$ and $U_1 \sim V_R + R^{-1} V_\theta$ as $R \to \infty$. Let $\phi = \Sigma A_n P_{\nu_n}(\mu) R^{\nu_n}$, where $\nu_n < \nu_{n+1}$ and $\mu = \cos\theta$, then

$$V_\theta \sim -A_1 P'_{\nu_1}(\mu) \sin\theta\, R^{\nu_1 - 1} \quad \text{as} \quad R \to \infty, \quad \mu \to -1.$$

Now as $\mu \to -1$,

$$P_\nu(\mu) \sim \pi^{-1} \sin\nu\pi\{\log(1+\mu) + \gamma + 2\psi(\nu+1) + \pi\cot\nu\pi - \log 2\}$$

and for a paraboloid we have

$$z \sim \tfrac{1}{2}r^2, \quad R \sim \tfrac{1}{2}r^2(1 + 2/r^2 + \ldots), \quad \cos\theta \sim -(1 - 2/r^2 + \ldots)$$

and $\sin\theta \sim 2/r$ as $r \to \infty$. Thus $V_\theta \sim -A_1\pi^{-1}\sin(\nu_1\pi)\, r(\tfrac{1}{2}r^2)^{\nu_1-1}$ as $r \to \infty$ and from §3 we know that $v_0 \sim -(\tfrac{5}{2})^{\frac{1}{4}} \bar{F}_{00}(\infty)\, r^{-\frac{1}{2}}$ as $r \to \infty$. It follows that $\nu_1 = \tfrac{1}{4}$,

$$A_1 = -\pi 2^{-\frac{3}{4}} \operatorname{cosec}(\nu_1\pi)\, (\tfrac{5}{2})^{\frac{1}{4}} \bar{F}_{00}(\infty)$$

and that

$$\begin{aligned} U_1 &\sim A_1 \nu_1 P_{\nu_1}(\mu) R^{\nu_1 - 1} - A_1\pi^{-1}\sin(\nu_1\pi)(\tfrac{1}{2}r^2)^{\nu_1-1} \\ &= A_1\pi^{-1}\sin(\nu_1\pi)(\tfrac{1}{2}r^2)^{\nu_1-1}\{\nu_1(\gamma + 2\psi(\nu_1+1) + \pi\cot(\nu_1\pi) - 2\log r) - 1\} \\ &= (\tfrac{5}{2})^{\frac{1}{4}} \bar{F}_{00}(\infty)\{1 + \tfrac{1}{4}(\log r^2 - \gamma - 2\psi(\tfrac{5}{4}) - \pi\}\, r^{-\frac{3}{2}}. \end{aligned} \tag{4.7}$$

This yields $U = 0{\cdot}136$ (compared with the computed value 0·107) for $r = 5$ and $U = 0{\cdot}061$ (compared with 0·055) for $r = 10$.

5. The second-order boundary layer

The pressure gradient in (2.12*a*) is obtained from (2.12*b*) as

$$\frac{\partial p_1}{\partial s} = -\frac{\partial}{\partial s}\left\{\kappa \int_n^\infty u_0^2\, dn\right\} + \frac{\partial}{\partial s}\left\{\cos\theta \int_n^\infty T_0\, dn\right\}. \tag{5.1}$$

We define the second-order stream function ψ_1 to satisfy (2.12*d*) identically, i.e.

$$\left.\begin{aligned} \psi_{1n} &= ru_1 + n\sin\theta\, u_0 - \bar{\chi} r u_0 T_0, \\ \psi_{1s} &= -rv_1 - v_0(rn\kappa + n\sin\theta - \bar{\chi} r T_0). \end{aligned}\right\} \tag{5.2}$$

Using (2.11), (5.2) and the definition of ψ_0, (2.12a, c) become

$$\left(\psi_{1n}\frac{\partial}{\partial s}-\psi_{1s}\frac{\partial}{\partial n}\right)\left(\frac{\psi_{0n}}{r}\right)+\left(\psi_{0n}\frac{\partial}{\partial s}-\psi_{0s}\frac{\partial}{\partial n}\right)\left(\frac{\psi_{1n}}{r}\right)-\psi_{1nnn}-r\sin\theta T_1$$
$$=-r\frac{\partial p_1}{\partial s}+\kappa r\left\{n\frac{\partial^2 u_0}{\partial n^2}+n\sin\theta T_0+\frac{\partial u_0}{\partial n}-u_0v_0\right\}+\sin\theta\left\{n\sin\theta T_0-\frac{\partial u_0}{\partial n}\right\}$$
$$+r\left(u_0\frac{\partial}{\partial s}+v_0\frac{\partial}{\partial n}\right)\left(\frac{n\sin\theta}{r}u_0\right)+\overline{\chi}\left\{\mu_1 T_0\frac{\partial^2 u_0}{\partial n^2}+(2+\mu_1)\frac{\partial T_0}{\partial n}\frac{\partial u_0}{\partial n}\right.$$
$$\left.+\left(1-\frac{1}{\sigma}\right)u_0\frac{\partial^2 T_0}{\partial n^2}-\sin\theta\, T_0^2+\beta_1 T_0^2\sin\theta\right\}, \tag{5.3}$$

$$\frac{1}{r}\left(\psi_{1n}\frac{\partial T_0}{\partial s}-\psi_{1s}\frac{\partial T_0}{\partial n}\right)+\frac{1}{r}\left(\psi_{0n}\frac{\partial T_1}{\partial s}-\psi_{0s}\frac{\partial T_1}{\partial n}\right)-\frac{1}{\sigma}\frac{\partial^2 T_1}{\partial n^2}$$
$$=\frac{1}{\sigma}\left\{\kappa+\frac{\sin\theta}{r}\right\}\left\{n\frac{\partial^2 T_0}{\partial n^2}+\frac{\partial T_0}{\partial n}\right\}+\overline{\chi}\frac{k_1}{\sigma}\left\{T_0\frac{\partial^2 T_0}{\partial n^2}+\left(\frac{\partial T_0}{\partial n}\right)^2\right\}. \tag{5.4}$$

These equations are linear in (ψ_1, T_1) and may be subdivided into a number of simpler problems. Of the terms on the right sides of (5.3) and (5.4) those in κ represent the effect of longitudinal curvature (l) and those in $(\sin\theta)/r$ arise from transverse curvature effects (t). Buoyancy effects are present in both terms and also couple the equations through the term $r\sin\theta T_1$ on the left side of (5.3). The second term in (5.1) is due to a heat flux (h) arising from the component of the body force along the surface and terms in $\overline{\chi}$ are due to the variation in the physical properties of the fluid with temperature. We distinguish those due to variation of density (dv) (a correction to the Boussinesq approximation), viscosity (vv), thermometric conductivity (tv) and the coefficient of expansion (cv). The boundary conditions are $\psi_1 = \psi_{1n} = T_1 = 0$ at $n = 0$ and the matching conditions give $u_1 \to U_0$ and $T_1 \to 0$ as $n \to \infty$. This means that we may distinguish the contribution of the displacement flow (d) as a specified tangential velocity at infinity. We thus define

$$\psi_1 = \psi_1^{(l)}+\psi_1^{(t)}+\psi_1^{(h)}+\overline{\chi}(\psi_1^{(dv)}+\mu_1\psi_1^{(vv)}+k_1\psi_1^{(tv)}+\beta_1\psi_1^{(cv)})+\psi_1^{(d)}, \tag{5.5}$$

and similarly for T_1.

We now define the operators Θ and Φ by

$$\left.\begin{aligned}\Theta(F,G) &\equiv \frac{\partial^3 F}{\partial n^3}-\left(\frac{\partial F}{\partial n}\frac{\partial u_0}{\partial s}-\frac{\partial F}{\partial s}\frac{\partial u_0}{\partial n}\right)-r\left(u_0\frac{\partial}{\partial s}+v_0\frac{\partial}{\partial n}\right)\left(\frac{1}{r}\frac{\partial F}{\partial n}\right)+r\sin\theta G,\\ \Phi(F,G) &\equiv \frac{1}{\sigma}\frac{\partial^2 G}{\partial n^2}-\frac{1}{r}\left(\frac{\partial T_0}{\partial s}\frac{\partial F}{\partial n}-\frac{\partial T_0}{\partial n}\frac{\partial F}{\partial s}\right)-\left(u_0\frac{\partial G}{\partial s}+v_0\frac{\partial G}{\partial n}\right).\end{aligned}\right\} \tag{5.6}$$

Then

$$\Theta(\psi_1^{(l)}, T_1^{(l)}) = -r\frac{\partial}{\partial s}\left\{\kappa\int_n^\infty u_0^2\,dn\right\}-r\kappa\left\{\frac{\partial u_0}{\partial n}-u_0v_0+n\frac{\partial^2 u_0}{\partial n^2}+n\sin\theta T_0\right\}, \tag{5.7a}$$

$$\Phi(\psi_1^{(l)}, T_1^{(l)}) = -(\kappa/\sigma)\,\partial\{n\,\partial T_0/\partial n\}/\partial n, \tag{5.7b}$$

$$\Theta(\psi_1^{(t)}, T_1^{(t)}) = -r\left(u_0\frac{\partial}{\partial s}+v_0\frac{\partial}{\partial n}\right)\left(\frac{\sin\theta}{r}nu_0\right)+\sin\theta\left\{\frac{\partial u_0}{\partial n}-n\sin\theta T_0\right\}, \tag{5.7c}$$

$$\Phi(\psi_1^{(t)},T_1^{(t)}) = -\frac{1}{\sigma}\frac{\sin\theta}{r}\frac{\partial}{\partial n}\left\{n\frac{\partial T_0}{\partial n}\right\}, \quad \Theta(\psi_1^{(h)},T_1^{(h)}) = r\frac{\partial}{\partial s}\left\{\int_n^\infty \cos\theta\, T_0\, dn\right\}, \tag{5.7d, e}$$

$$\Theta(\psi_1^{(dv)},T_1^{(dv)}) = -\left\{2r\frac{\partial T_0}{\partial n}\frac{\partial u_0}{\partial n}+\left(1-\frac{1}{\sigma}\right)ru_0\frac{\partial^2 T_0}{\partial n^2}-r\sin\theta\, T_0^2\right\}, \tag{5.7f}$$

$$\Theta(\psi_1^{(vv)},T_1^{(vv)}) = -r\left\{T_0\frac{\partial^2 u_0}{\partial n^2}+\frac{\partial T_0}{\partial n}\frac{\partial u_0}{\partial n}\right\}, \quad \Phi(\psi_1^{(tv)},T_1^{(tv)}) = -\frac{1}{\sigma}\left\{T_0\frac{\partial^2 T_0}{\partial n^2}+\left(\frac{\partial T_0}{\partial n}\right)^2\right\}, \tag{5.7g, h}$$

$$\Theta(\psi_1^{(cv)},T_1^{(cv)}) = -r\sin\theta\, T_0^2, \tag{5.7i}$$

$$\Phi(\psi_1^{(h)},T_1^{(h)}) = \Phi(\psi_1^{(dv)},T_1^{(dv)}) = \Phi(\psi_1^{(vv)},T_1^{(vv)}) = 0, \tag{5.7j}$$

$$\Theta(\psi_1^{(tv)},T_1^{(tv)}) = \Phi(\psi_1^{(cv)},T_1^{(cv)}) = 0, \quad \Theta(\psi_1^{(d)},T_1^{(d)}) = \Phi(\psi_1^{(d)},T_1^{(d)}) = 0. \tag{5.7k, l}$$

The boundary conditions for all components are

$$\psi_1 = \psi_{1n} = T_1 = 0 \quad \text{at} \quad n = 0; \quad \psi_{1n} \to 0, \quad T_1 \to 0 \quad \text{as} \quad n \to \infty$$

except for $\psi_1^{(d)}$, which satisfies $\psi_{1n}^{(d)} \to rU_1$ as $n \to \infty$.

After application of the Saville–Churchill transformation equations (5.6) and (5.7) become

$$\left.\begin{aligned}\Theta(F_1,T_1) &\equiv F_1'''+T_1-\tfrac{4}{3}\xi\{(F_{0\xi}F_1'-F_0''F_{1\xi})+(F_0'F_{1\xi}'-F_1''F_{0\xi})\}\\ &\qquad +F_0''F_1+F_0F_1''-\tfrac{8}{3}K(\xi)F_0'F_1',\\ \Phi(F_1,T_1) &\equiv \sigma^{-1}T_1''-\tfrac{4}{3}\xi\{(T_{0\xi}F_1'-T_0'F_{1\xi})+(F_0'T_{1\xi}-F_{0\xi}T_1')\}+T_0'F_1+F_0T_1',\end{aligned}\right\} \tag{5.8}$$

$$\Theta(F_1^{(l)},T_1^{(l)}) = L_1(\xi)\{F_0''+\eta F_0'''+\eta T_0+F_0F_0'+\tfrac{4}{3}\xi F_0'F_{0\xi}\}+L_2(\xi)\int_\eta^\infty F_0'^2\,d\eta, \tag{5.9a}$$

$$\Phi(F_1^{(l)},T_1^{(l)}) = L_1(\xi)\,\sigma^{-1}\{T_0'+\eta T_0''\}, \tag{5.9b}$$

$$\Theta(F_1^{(t)},T_1^{(t)}) = T_1(\xi)\{\tfrac{4}{3}\xi\eta(F_0'F_{0\xi}'-F_0''F_{0\xi})-\tfrac{4}{3}\xi F_0'F_{0\xi}-\eta F_0F_0''-F_0F_0'-F_0''+\eta T_0\} + T_2(\xi)\,\eta F_0'^2, \tag{5.9c}$$

$$\Phi(F_1^{(t)},T_1^{(t)}) = \sigma^{-1}T_1(\xi)\{T_0'+\eta T_0''\}, \tag{5.9d}$$

$$\Theta(F_1^{(h)},T_1^{(h)}) = H_1(\xi)\,\eta T_0+H_2(\xi)\int_\eta^\infty T_0\,d\eta+H_3(\xi)\int_\eta^\infty T_{0\xi}\,d\eta, \tag{5.9e}$$

$$\Theta(F_1^{(dv)},T_1^{(dv)}) = (\sigma^{-1}-1)F_0'T_0''-2T_0'F_0''-T_0^2, \tag{5.9f}$$

$$\Theta(F_1^{(vv)},T_1^{(vv)}) = -\{T_0F_0'''+T_0'F_0''\}, \quad \Phi(F_1^{(tv)},T_1^{(tv)}) = -\sigma^{-1}\{T_0T_0''+T_0'^2\}, \tag{5.9g, h}$$

$$\Theta(F_1^{(cv)},T_1^{(cv)}) = -T_0^2, \tag{5.9i}$$

$$\Phi(F_1^{(h)},T_1^{(h)}) = \Phi(F_1^{(dv)},T_1^{(dv)}) = \Phi(F_1^{(vv)},T_1^{(vv)}) = 0, \tag{5.9j}$$

$$\Theta(F_1^{(tv)},T_1^{(tv)}) = \Phi(F_1^{(cv)},T_1^{(cv)}) = 0, \quad \Theta(F_1^{(d)},T_1^{(d)}) = \Phi(F_1^{(d)},T_1^{(d)}) = 0, \tag{5.9k, l}$$

and the boundary conditions for all components are $F_1 = F_1' = T_1 = 0$ at $\eta = 0$ and $F_1' \to 0$ and $T_1 \to 0$ as $\eta \to \infty$ except for $F_1^{(d)}$, which satisfies $F_1^{(d)\prime} \to D(\xi$ as $\eta \to \infty$. The shape of the body enters these equations only through the 'secondary' functions

$$L_1(\xi) = -\frac{K}{rZ}\left(\frac{4\xi}{3}\right)^{\frac{1}{4}}, \quad L_2(\xi) = -\frac{d}{d\xi}\left[\frac{KZ}{r}\left(\frac{4\xi}{3}\right)^{\frac{5}{4}}\right]\frac{1}{Z^2},$$

$$T_1(\xi) = -(\tfrac{4}{3}\xi)^{\frac{1}{2}}Z^2, \quad T_2(\xi) = T_1(\xi)\,(1+4\xi Z'/Z),$$

$$H_1(\xi) = -\left\{\frac{dr}{ds} + r^3\left(Z' - \frac{Z}{4\xi}\right)\right\}\left(\frac{4\xi}{3}\right)^{\frac{1}{4}}\frac{\cos\theta}{r^4Z^4},$$

$$H_2(\xi) = \frac{d}{d\xi}\left\{\left(\frac{4\xi}{3}\right)^{\frac{1}{4}}\frac{\cos\theta}{rZ^3}\right\}\frac{1}{Z^2}, \quad H_3(\xi) = \left(\frac{4\xi}{3}\right)^{\frac{1}{4}}\frac{\cos\theta}{rZ^3},$$

$$D(\xi) = (\tfrac{4}{3}\xi)^{-\frac{1}{2}}\, r^{\frac{1}{3}}\,(1+r^2)^{\frac{1}{6}}\, U_1(\xi),$$

where $Z = [(\sin\theta)/r]^{\frac{1}{3}}$, which are of fundamental importance in determining the nature of the secondary flow. The introduction of $L_2(\xi)$ as well as $L_1(\xi)$ and so on is a reflexion of the incompleteness of our subdivision (5.5) of (5.3) and (5.4), but any further subdivision would be unnecessarily cumbersome. For a paraboloid these functions are

$$L_1(\xi) = -(\tfrac{4}{3}\xi)^{\frac{1}{4}}\, r^{-\frac{2}{3}}(1+r^2)^{-\frac{4}{3}}, \quad L_2(\xi) = \tfrac{4}{3}\xi\, L_1' + \tfrac{8}{3}L_1\, K(\xi),$$

$$T_1(\xi) = -(\tfrac{4}{3}\xi)^{\frac{1}{4}}\, r^{-\frac{2}{3}}\,(1+r^2)^{-\frac{1}{3}}, \quad T_2(\xi) = T_1(\xi)\,(4K(\xi)-1),$$

$$H_1(\xi) = -(\tfrac{4}{3}\xi)^{\frac{1}{4}}\,[r^{-\frac{8}{3}}(1+r^2)^{-\frac{1}{3}} + (K(\xi)-\tfrac{3}{4})/\xi],$$

$$H_2(\xi) = (\tfrac{4}{3}\xi)^{-\frac{3}{4}}\,(\tfrac{8}{3}K(\xi)-1), \quad H_3(\xi) = (\tfrac{4}{3}\xi)^{\frac{1}{4}}.$$

Expanding for $\xi \ll 1$ we get

$$L_1 \sim L_2 \sim T_1 \sim 2T_2 \sim -\tfrac{7}{3}H_1 \sim \tfrac{7}{4}H_2 \sim -2^{-\frac{1}{4}},$$

$$H_3\int_\eta^\infty T_{0\xi}\,d\eta \sim \int_\eta^\infty T_{01}\,d\eta, \quad D \sim 0{\cdot}6331 \quad (\text{for} \quad \sigma = 1)$$

and the first terms in the expansions of F_1 and T_1 about $\xi = 0$ are obtained by substituting these values into (5.9). For $\xi \gg 1$ we find that

$$L_1(\xi) \sim -(\tfrac{2}{5})\,(\tfrac{4}{3}\xi)^{-\frac{3}{4}}, \quad L_2(\xi) \sim \tfrac{1}{5}(\tfrac{2}{5})\,(\tfrac{4}{3}\xi)^{-\frac{3}{4}},$$

$$T_1(\xi) \sim -(\tfrac{2}{5})^{\frac{1}{4}}\,(\tfrac{10}{3}\xi)^{-\frac{3}{20}}, \quad T_2(\xi) \sim -\tfrac{1}{5}(\tfrac{2}{5})^{\frac{1}{4}}\,(\tfrac{10}{3}\xi)^{-\frac{3}{20}},$$

$$H_1(\xi) \sim \tfrac{1}{5}(\tfrac{4}{3}\xi)^{-\frac{3}{4}}, \quad H_2(\xi) \sim -\tfrac{1}{5}(\tfrac{4}{3}\xi)^{-\frac{3}{4}},$$

$$H_3(\xi)\int_\eta^\infty T_{0\xi}\,d\eta \sim -\tfrac{8}{15}(\tfrac{4}{3}\xi)^{-\frac{7}{4}}\int_\eta^\infty \overline{T}_{01}\,d\eta,$$

$$D(\xi) \sim \overline{F}_{00}(\infty)\,(\tfrac{4}{3}\xi)^{-\frac{3}{4}}\{1 + \tfrac{3}{20}\log(\tfrac{10}{3}\xi) - \tfrac{1}{4}(\gamma + 2\psi(\tfrac{5}{4}) + \pi)\},$$

the latter being obtained by use of (4.7). This suggests that we expand F_1 in inverse powers of ξ as

$$F_1^{(l)} = (\tfrac{2}{5})\,(\tfrac{4}{3}\xi)^{-\frac{3}{4}}\{\overline{F}_{10}^{(l)} + \ldots\},$$

$$F_1^{(t)} = (\tfrac{2}{5})^{\frac{1}{4}}\,(\tfrac{10}{3}\xi)^{-\frac{3}{20}}\{\overline{F}_{10}^{(t)} + \ldots\},$$

$$F_1^{(h)} = \tfrac{1}{5}(\tfrac{4}{3}\xi)^{-\frac{3}{4}}\{\overline{F}_{10}^{(n)} + \ldots\},$$

$$F_1^{(v)} = \overline{F}_{10}^{(v)} + \ldots,$$

$$F_1^{(d)} = \overline{F}_{00}(\infty)\{(\tfrac{4}{3}\xi)^{-\frac{3}{4}}\log\xi\,\overline{F}_{10}^{(d)} + a_1(\tfrac{4}{3}\xi)^{-\frac{3}{4}}\,\overline{F}_{11}^{(d)} + \ldots\},$$

and similarly for T, where (v) denotes (dv), (vv), (tv) or (cv).

The derivation of the ordinary differential equations satisfied by these coefficients has been omitted for the sake of brevity, but details are given in the longer version of §5 mentioned in the footnote to §3 which is available from the J.F.M. Editorial Office.

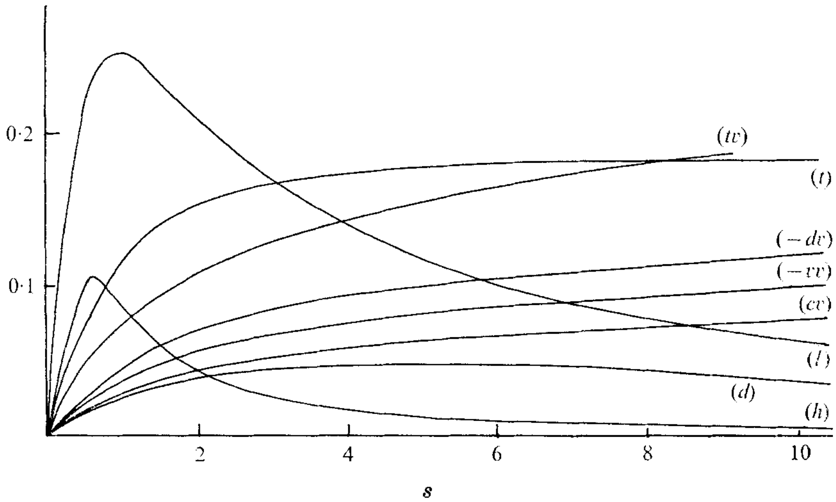

FIGURE 1. Second-order skin-friction parameter $\partial u_1/\partial n$ for $\sigma = 1$. The (dv), (vv) and (cv) components have been reduced by a factor of 10.

As in §3.1 we adopt Merk's method in the region where neither series holds, the only novel feature being the removal of the secondary functions, $L_1(\xi)$, etc., as factors before integrating.

6. Results and conclusions

The numerical integration of the systems of ordinary differential equations obtained in §§3 and 5 has been performed for $\sigma = 0{\cdot}5$, $1{\cdot}0$ and $2{\cdot}0$ using a 'shooting' technique and all the relevant starting values are tabulated in Walton (1972).

The skin friction τ and local Nusselt number N are defined by

$$\frac{\tau}{\rho U_0^2} = G^{-\frac{1}{4}}\frac{\partial u}{\partial n}(s,0) = G^{-\frac{1}{4}}\left\{\frac{\partial u_0}{\partial n}(s,0) + G^{-\frac{1}{4}}\frac{\partial u_1}{\partial n}(s,0) + \dots\right\},$$

$$N = \frac{lq}{k(T_w - T_0)} = -G^{-\frac{1}{4}}\frac{\partial T}{\partial n}(s,0) = -G^{-\frac{1}{4}}\left\{\frac{\partial T_0}{\partial n}(s,0) + G^{-\frac{1}{4}}\frac{\partial T_1}{\partial n}(s,0) + \dots\right\},$$

where q is the heat transfer from the wall. In Saville–Churchill co-ordinates

$$\partial u(s,0)/\partial n = r^{\frac{1}{3}}(1+r^2)^{-\frac{1}{3}}\left(\tfrac{4}{3}\xi\right)^{\frac{1}{4}}F''(\xi,0),$$

$$\partial T(s,0)/\partial n = r^{\frac{2}{3}}(1+r^2)^{-\frac{1}{6}}\left(\tfrac{4}{3}\xi\right)^{-\frac{1}{4}}T'(\xi,0).$$

Values of $\partial u(s,0)/\partial n$ and $\partial T(s,0)/\partial n$ are tabulated in Walton (1972) and are illustrated in figures 1 and 2.

It can be seen that in the region of the stagnation point longitudinal curvature tends to decrease the skin friction, transverse curvature to increase it and displacement effects to decrease it. Similar effects for isothermal flow were found by Tani (1954), Eshghy & Hornbeck (1967) and Van Dyke (1962*b*), and Van Dyke

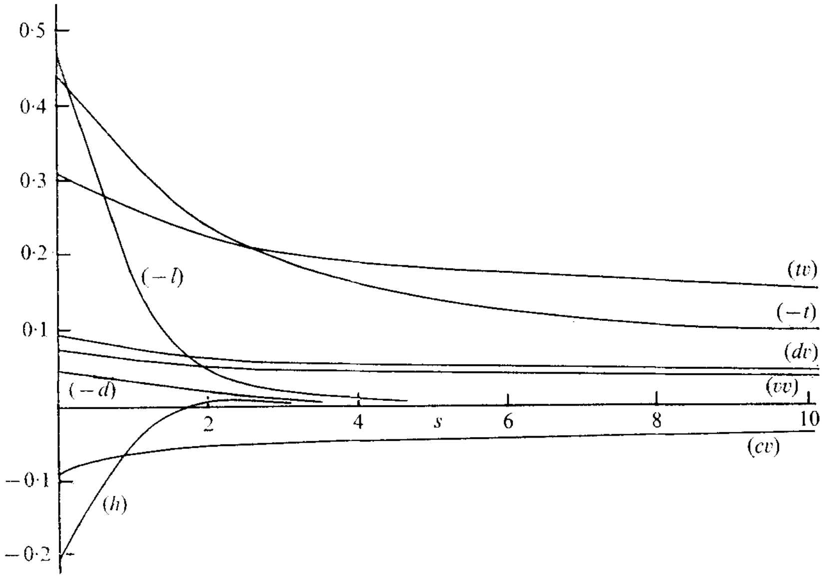

FIGURE 2. Second-order heat-transfer parameter $\partial T_1/\partial n$ for $\sigma = 1$. The (h) component has been magnified by a factor of 10.

(1962*a*) respectively and the result due to displacement effects is in accord with the results of Yang & Jerger (1964) for free convection from a vertical flat plate.

Other important quantities are the flux thicknesses defined as follows:

$$\delta_1 = \int_0^\infty u^2\,dn, \quad \delta_2 = \int_0^\infty T\,dn, \quad \delta_3 = \int_0^\infty uT\,dn,$$

which we may write as

$$\delta_i = \delta_{i0} + G^{-\frac{1}{4}}\delta_{i1} + \ldots \quad (i = 1, 2, 3),$$

where

$$\delta_{10} = \int_0^\infty u_0^2\,dn, \quad \delta_{20} = \int_0^\infty T_0\,dn, \quad \delta_{30} = \int_0^\infty u_0 T_0\,dn,$$

$$\delta_{11} = 2\int_0^\infty u_0 u_1\,dn, \quad \delta_{21} = \int_0^\infty T_1\,dn, \quad \delta_{31} = \int_0^\infty (u_0 T_1 + u_1 T_0)\,dn.$$

These quantities are also tabulated in Walton (1972).

For gases typical values of β_1, μ_1 and k_1 are -1, 0·6 and 0·6 respectively and χ and $G^{-\frac{1}{4}}$ are of the same order, so that figures 1 and 2 closely resemble the actual ones. For liquids, however, $\chi \ll G^{-\frac{1}{4}}$ and those curves representing the variation of physical properties should be ignored.

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